Multiple-Antenna Interference Network with Receive Antenna Joint Processing and Real Interference Alignment

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Abstract

In this paper, the degrees of freedom (DoF) regions of constant coefficient multiple antenna interference channels are investigated. First, we consider a K-user Gaussian interference channel with M_k antennas at transmitter k, $1 \leq k \leq K$, and N_j antennas at receiver j, $1 \leq j \leq K$, denoted as a $(K, [M_k], [N_j])$ channel. Relying on a result of simultaneous Diophantine approximation, a real interference alignment scheme with joint receive antenna processing is developed. The scheme is used to obtain an achievable DoF region. The proposed DoF region includes two previously known results as special cases, namely 1) the total DoF of a K-user interference channel with N antennas at each node, $(K, [N], [N])$ channel, is $NK/2$; and 2) the total DoF of a $(K, [M], [N])$ channel is at least $KMN/(M + N)$. We next explore constant coefficient interference networks with K transmitters and J receivers, all having N antennas. Each transmitter emits an independent message and each receiver requests an arbitrary subset of the messages. Employing the novel joint receive antenna processing, the DoF region for this set-up is obtained. We finally consider wireless X networks where each node is allowed to have an arbitrary number of antennas. It is shown that the joint receive antenna processing can be used to establish an achievable DoF region, which is larger than what is possible with antenna splitting. As a special case of the derived achievable DoF region for constant coefficient X network, the total DoF of wireless X networks with the same number of antennas at each node is tight in certain cases while the best inner bound based on antenna splitting cannot meet the outer bound.

I. Introduction

Characterizing the capacity region of interference networks is a fundamental communication problem. This area has seen extensive activities leading to remarkable progress on a variety of problems including the capacity region of interference channels, X networks, and cellular networks. Theses efforts have produced results that shed light on many aspects of the problems. Nevertheless, the capacity region of interference networks still remains unknown in general.

Recent work has proposed to use DoF to approximate the capacity region of interference networks. The DoF of a message is its rate normalized by the capacity of single-user additive white Gaussian noise channel, as the SNR tends to infinity. The DoF region quantifies the shape of the capacity region at high signal-to-noise ratio (SNR) [1], [2].

DoF investigations have motivated several fundamental ideas such as interference alignment and rational dimensions. Interference alignment is an essential approach for achieving the optimal DoF. It proposes that at a receiver, the interference signals from multiple transmitters be aligned in the signal space, so that the dimensionality of the interference in the signal space can be minimized. Therefore, the remaining space is interference free and can be used for the desired signals. Two commonly used alignment schemes are vector alignment and real alignment [3], [4].

- In vector alignment, any transmit signal is a linear combination of some vectors in a manner that the coefficients of the linear combination carry useful data. This scheme designs the vectors such that the interferences at each receiver are squeezed into a common subspace of the available signal space. The orthogonal complement can be used for detecting useful data symbols.
- In real alignment, the concept of linear independence over the rational numbers replaces the more familiar vector linear independence. And a Groshev type theorem is usually used to guarantee the required decoding performance.

A. DoF of interference channel

DoF characterizations have been obtained for a variety of wireless networks such as K-user interference channel and wireless X network. In the K-user interference, the k-th transmitter has a message intended for the k-th receiver. At receiver k, the messages from transmitters other than the k-th are interference. The DoF region of the K-user interference when all nodes are provided with the same number of antennas is known [5].

In [6], Gou and Jafar studied the total DoF of the $M \times N$ K-user interference channel where each transmitter has M antennas and each receiver has N antennas. They showed the exact total DoF value is $K\frac{MN}{M+N}$ under the assumption that $R = \frac{\max(M,N)}{\min(M,N)}$ is an integer and $K \geq R$. In [7], Ghasemi et al. employ antenna splitting argument to derive the total DoF $K\frac{MN}{M+N}$ for fixed channels, which is optimal if $K \geq \frac{M+N}{\gcd(M,N)}$ even when R is not an integer. In such antenna splitting arguments, no cooperation is used either at the transmitter side or at the receiver side. The outer bounds of these cases are based on cooperation among groups of transmitters and receivers and employing the DoF outer bound for 2-user multiple-input multiple-output (MIMO) interference channel obtained in [8]. Note that the outer bound discussion is regardless of whether the channel coefficients are constant or time-varying.

A novel genie chains approach for the DoF outer bound of $M \times N$ K-user interference channel has been recently presented in [9]. In this approach, a chain of mapping from genie signals provided at a receiver to the exposed signal spaces of the receiver is served as the genie signals for the next receiver until a genie with an acceptable number of dimensions is obtained. As a result, it is proved that for any $K \geq 4$, the total DoF is outer bounded by $K\frac{MN}{M+N}$ as long as $R \geq \frac{K-2}{K^2-3K+1}$.

The DoF region of MIMO K-user interference channels has not been accomplished in general for arbitrary number of antennas except for the 2-user case [8].

B. DoF of X network

There is also increasing interest in characterizing DoF region of MIMO X networks. A $K \times J$ user MIMO X network consists of K transmitters and J receivers where each transmitter has an independent message for each receiver. Notably, the X networks include interference channels as a special case.

The previously best known inner bounds on the total DoF of $K \times J$ user MIMO time-varying X networks with N antennas at each node are based on

1) Antenna splitting with no cooperation [10]: The achievable total DoF is attained by decomposing all transmitter and receiver antennas in which we have an $NK \times NJ$ user single-input single-output (SISO) X network. Therefore, the best total DoF $N\frac{KJ}{K+J-\frac{1}{N}}$ is achieved. However, there is a gap between the inner bound and the DoF outer bound, $N\frac{KJ}{K+J-1}$, implying that a cooperation structure might be needed here.
2) Joint signal processing [11]: Doing joint processing at either transmitter or receiver side, the desired signals at any receiver can be efficiently resolved from the interference. This new insight closes the mentioned gap and so the total DoF value $N\frac{KJ}{K+J-1}$ is achieved.

These results offer an opportunity to revise our understanding of antenna splitting technique. In fact, independent processing at each antenna was first made to simplify the achievability scheme of K-user MIMO interference channels, which turned out to be optimal in some cases. However, [11] shows that allowing cooperation between antennas is essential for establishing the desired DoF.

In the class of real interference alignment, the DoF of $K \times J$ user MIMO X networks has not been studied to the best of our knowledge. Also, except for the two-user case [12], the DoF region of MIMO X networks when each node has an arbitrary number of antennas has not been considered yet.

C. Summary of results

In this paper, the results from the field of Diophantine approximation for systems of m linear forms in n variables are used to establish a new joint receive antenna processing. Employing this novel procedure, we characterize the DoF region of some classes of time-invariant multiple antenna interference networks.

To introduce the main concepts, we first study a constant K-user MIMO Gaussian interference channel with N antennas at each node. Relying on the recent results on simultaneous Diophantine approximation, we develop a real interference alignment scheme for this channel to introduce our new joint receive antenna processing. We obtain alternative proofs of several previously known results; see Section IV. Next, we focus on K-user MIMO Gaussian interference equipped with M antennas at each transmitter and N antennas at each receiver. For this scenario, an achievable DoF region is attained. It is shown that the achieved DoF region includes the previously known results as special cases. We also establish an achievable DoF region for the K-user MIMO Gaussian interference when each node has an arbitrary number of antennas.

We then consider $K \times J$ user MIMO interference network with general message demands under assumption that all nodes have the same number of antennas. In this model, each transmitter conveys an independent message and each receiver requests an arbitrary subset of messages. Utilizing joint receive antenna processing, the exact DoF region is established.

We finally apply our new scheme to the $K \times J$ user MIMO X network and derive an achievable DoF region which is shown to be tight under certain circumstances.

In summery, the key to these results is our novel joint receive antenna processing that can be applied to many interference network models. We begin with an outline of the new approach.

II. DIOPHANTINE APPROXIMATION AND JOINT RECEIVE ANTENNA PROCESSING

Notation: Throughout we use $\|\mathbf{x}\|$ to denote the infinity norm of vector $\mathbf{x}$ and $\cdot$ the inner product of two vectors. The set of integers, positive integers, and real numbers are denoted as $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$, respectively.

A. Diophantine approximation

Diophantine approximation is to approximate real numbers by rational numbers. The original problem of this area was to know how good the approximation can be under certain constraints. For this problem, a rational number $\frac{a}{b}$ is a good approximation of a real number ω, if $|\omega - \frac{a}{b}|$ does not increase when $\frac{a}{b}$ is replaced by another rational number with smaller denominator. A major challenge is to figure out sharp upper and lower bounds of $|\omega - \frac{a}{b}|$, formulated as a function of denominator b. To introduce some recent results of Diophantine approximation, we need the notion of rational independence.

Definition 1: A set of real numbers is *rationally independent* if none of the elements of the set can be written as a linear combination of the other elements with rational coefficients.

Let $\{\omega_1, \ldots, \omega_n\}$ be a set of rationally independent numbers. The rational independence is to guarantee that the absolute value of any linear combination of $\{\omega_1, \ldots, \omega_n\}$ with rational coefficients is strictly greater than zero unless all coefficients are zero. We call n the rational dimension of the introduced set. In this setup, the metric Diophantine approximation deals with approximation properties of the vector of real numbers $\boldsymbol{\omega} := (\omega_1, \ldots, \omega_n)$ by a vector of rational numbers $(\frac{a_1}{b}, \ldots, \frac{a_n}{b})$ all having the same denominator b. In particular, the problem of finding a sharp lower bound on $\|\boldsymbol{\omega} \cdot \mathbf{q}\|$, for all non-zero $\mathbf{q} \in \mathbb{Z}^n$, is the main concern of Diophantine approximation. Khintchine's theorem and its generalization by Groshev can be mentioned as the well-done works here. In a special case of Khintchine-Groshev theorem, we have the following theorem [13], [14].

Theorem 1: For almost all $\boldsymbol{\omega} \in \mathbb{R}^n$ in the Lebesgue sense, there exists a positive constant κ such that the system of inequality

$$\|\boldsymbol{\omega} \cdot \mathbf{q} + p\| > \frac{\kappa}{\|\mathbf{q}\|^{n+\epsilon}}, \quad \epsilon > 0 \tag{1}$$

holds for all non-zero $\mathbf{q} \in \mathbb{Z}^n$ and all $p \in \mathbb{Z}$.

The above theorem applies to the case where the elements of $\boldsymbol{\omega}$ are chosen independently of each other. If the point $\boldsymbol{\omega}$ is restricted to lie on a submanifold of $\mathbb{R}^n$ of a dimension smaller than n, then the above theorem cannot be used to quantify the measure of well-approximable $\boldsymbol{\omega}$ on the submanifold. Such quantification is necessary, however, when studying interference channels.

B. Nondegenerate manifolds

One important notion in studying Diophantine approximation on manifolds is the non-degeneracy, which we introduce next [15], [16].

Definition 2: A set of functions are *linearly independent over* $\mathbb{R}$ if none of the functions can be represented by a linear combination of the other functions with real coefficients.

Definition 3 (Non-degenerate Manifolds): Consider a d-dimensional sub-manifold $\mathcal{M} = \{\mathbf{f}(\mathbf{x}) \,\big|\, \mathbf{x} \in U\}$ of $\mathbb{R}^n$, where $U \subset \mathbb{R}^d$ is an open set and $\mathbf{f} = (f_1, \ldots, f_n)$ is a C^k embedding of U to $\mathbb{R}^n$. For $l \leq k$, $\mathbf{f}(\mathbf{x})$ is an l-*non-degenerate* point of $\mathcal{M}$ if partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ of order up to l span the space $\mathbb{R}^n$. The function $\mathbf{f}$ at $\mathbf{x}$ is *non-degenerate* when it is l-nondegenerate at $\mathbf{x}$ for some l.

Fact [15]: If the functions $f_1, \ldots, f_n$ are analytic, and $1, f_1, \ldots, f_n$ are linearly independent over $\mathbb{R}$ in a domain U, all points of $\mathcal{M} = \mathbf{f}(U)$ are nondegenerate.

C. Diophantine approximation on manifolds

There has been a growing interest in Diophantine approximation of non-degenerate manifolds. Beresnevich extended Theorem 1 to the class of non-degenerate manifolds [16]. Interestingly in [4], Motahari et al. showed that if $\boldsymbol{\omega}$ lies on a non-degenerate manifold, it is sufficient to achieve interference alignment at the receivers. In this setup, the following Groshev type of theorem for convergence on manifolds of [16] was used.

Theorem 2: Let $n \geq 2$, $d \in \mathbb{N}$ with $d \leq n$, $U \subset \mathbb{R}^d$ be an open set and $\mathbf{f} := (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ such that all f_i, $1 \leq i \leq n$, are analytic and together with 1 linearly independent over $\mathbb{R}$ (hence $\mathbf{f}$ is nondegenerate). Then, there exists a non-zero constant κ in which for almost all $\mathbf{x} \in U$

$$\|\mathbf{f}(\mathbf{x}) \cdot \mathbf{q} + p\| > \frac{\kappa}{\|\mathbf{q}\|^{n+\epsilon}}, \quad \epsilon > 0 \tag{2}$$

holds for all non-zero $\mathbf{q} \in \mathbb{Z}^n$ and all $p \in \mathbb{Z}$. $\square$

In this paper, we propose a new joint receive antenna processing that requires simultaneous Diophantine approximation for the set-up of systems of m linear forms in n variables. To establish this scheme, we will need a theorem proved in [17].

Theorem 3: Let $\mathbf{f}_i$, $i = 1, \ldots, m$ be a non-degenerate map from an open set $U_i \subset \mathbb{R}^{d_i}$ to $\mathbb{R}^n$ and

$$F : U_1 \times \ldots \times U_m \to \mathcal{M}_{m,n}, \quad (\mathbf{x}_1, \ldots, \mathbf{x}_m) \longmapsto \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1) \\ \vdots \\ \mathbf{f}_m(\mathbf{x}_m) \end{pmatrix}$$

where $\mathcal{M}_{m,n}$ denotes the space of $m \times n$ real matrices.

Then, for $\epsilon > 0$, the set of $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ such that for

$$\mathbf{A} = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1) \\ \vdots \\ \mathbf{f}_m(\mathbf{x}_m) \end{pmatrix} \tag{3}$$

there exist infinitely many $\mathbf{q} \in \mathbb{Z}^n$ with

$$\|\mathbf{A}\mathbf{q} - \mathbf{p}\| < \|\mathbf{q}\|^{-\frac{n}{m}-\epsilon} \quad \text{for some } \mathbf{p} \in \mathbb{Z}^m \tag{4}$$

has zero Lebesgue measure on $U_1 \times \ldots \times U_m$. $\square$

The theorem states that almost surely for any fixed $\mathbf{A}$, there is a positive constant κ such that $\|\mathbf{A}\mathbf{q}\| > \kappa\|\mathbf{q}\|^{-\frac{n}{m}-\epsilon}$ holds for all non-zero $\mathbf{q}$. Here, m can be interpreted as the total number of antennas at a receiver with n available directions (will be specified later). The i-th row of $\mathbf{A}$, $\mathbf{f}_i$, is also interpreted as the received directions of antenna i at the receiver. Knowing all of these, we are able to derive the lower bound on the minimum distance of received constellation when all receiver antennas are cooperative.

We show in Section V that dimensions of matrix $\mathbf{A}$ play an important role in determining the DoF. We will see that achievability for DoF regions can be established once we construct a suitable matrix $\mathbf{A}$. As it turns out, Theorem 3 results an achievable DoF per direction of $\frac{m}{n}$. The concept of direction will be made more precise in Section V.

III. System model

Notation: K, J, M, N, D, and D' are integers and $\mathcal{K} = \{1, \ldots, K\}$, $\mathcal{J} = \{1, \ldots, J\}$, $\mathcal{M} = \{1, \ldots, M\}$, $\mathcal{N} = \{1, \ldots, N\}$. We use k, $\hat{k}$ as transmitter indices, and j, $\hat{j}$ as receiver indices. Superscripts t and r are used for transmitter and receiver antenna indices. The set of non-negative real numbers is denoted as $\mathbb{R}_+$. Letters i and l are used as the indices of directions and streams, respectively. Vectors and matrices are indicated by bold symbols. We use $[M_k]_{k=1}^{K}$ to denote vector $(M_1, \ldots, M_K)$, and $[d_{j,k}]_{j=1,k=1}^{J,K}$ the $J \times K$ matrix with element $d_{j,k}$ in the (j,k)th position. When there is no confusion, $[M_k]$ is used as an abbreviation for $[M_k]_{k=1}^{K}$, and $[M]$ is used to denote a vector where all M_k are equal to M. We also use $(\cdot)^*$ to denote matrix transpose, $\otimes$ the Kronecker product of two matrices, and $\cup$ union of sets.

Consider a MIMO real Gaussian interference network with K transmitters and J receivers. Suppose transmitter k has M_k antennas and receiver j has N_j antennas. At each time, each transmitter, say

transmitter k, sends a vector signal $\mathbf{x}_k \in \mathbb{R}^{M_k}$. The channel from transmitter k to receiver j is represented as a matrix

$$\mathbf{H}_{j,k} := [h_{j,k,r,t}]_{r=1,t=1}^{N_j,M_k} \tag{5}$$

where $k \in \mathcal{K}$, $j \in \mathcal{J}$, and $\mathbf{H}_{j,k} \in \mathbb{R}^{N_j \times M_k}$. It is assumed that the channel is constant during all transmissions. Each transmit antenna is subjected to an average power constraint P. The received signal at receiver j can be expressed as

$$\mathbf{y}_j = \sum_{k \in \mathcal{K}} \mathbf{H}_{j,k}\mathbf{x}_k + \boldsymbol{\nu}_j, \quad \forall j \in \mathcal{J} \tag{6}$$

where $\{\boldsymbol{\nu}_j | j \in \mathcal{J}\}$ is the set of independent Gaussian additive noises with real, zero mean, independent, and unit variance entries. Let $\mathbf{H}$ denote the $\sum_{j\in\mathcal{J}} N_j \times \sum_{k\in\mathcal{K}} M_k$ block matrix, whose (j,k)th block of size $N_j \times M_k$ is the matrix $\mathbf{H}_{j,k}$. The matrix $\mathbf{H}$ includes all the channel coefficients.

In view of message demands at receivers, the introduced channel can be specialize to three known cases:

1) *The $(K, J, [M_k], [N_j], [W_j])$ interference network with general message demands*: where each receiver, for instance receiver j, requests an arbitrary subsets of transmitted signals as $\mathcal{W}_j = \{k \in \mathcal{K} \,\big|\, \text{receiver } j \text{ requests } \mathbf{x}_k\}$.
2) *The single hop $(K, J, [M_k], [N_j])$ wireless X network*: where for each pair $(j,k) \in \mathcal{J} \times \mathcal{K}$, transmitter k conveys an independent message $\mathbf{x}_{k,j}$ to receiver j noting that $\mathbf{x}_k = \sum_{j\in\mathcal{J}} \mathbf{x}_{k,j}$.
3) *The K-user interference channel*: where $J = K$ and signal $\mathbf{x}_k$, $\forall k \in \mathcal{K}$, is just intended for receiver k. For this model, we use the abbreviation $(K, [M_k], [N_j])$.

In the case of K-user interference channel, the *capacity region* $\mathcal{C}_{IC}(P, K, [M_k], [N_j], \mathbf{H})$ is defined in the usual sense: It contains rate tuples $[R_k(P)]_{k=1}^K$ such that reliable transmission from transmitter k to receiver k is possible at rate R_k for all $k \in \mathcal{K}$ simultaneously, under the given power constraint P. Reliable transmissions mean that the probability of error can be made arbitrarily small by increasing the encoding block length while keeping the rates and power fixed.

A DoF vector $[d_k]_{k=1}^K$ is said to be *achievable* if for any large enough P, the rates $R_i = 0.5\log(P)d_i$, $i = 1, 2, \ldots, K$, are simultaneously achievable by all K users, namely $0.5\log(P) \cdot [d_k]_{k=1}^K \in \mathcal{C}_{IC}(P, K, [M_k], [N_j], \mathbf{H})$, for P large enough. The *DoF region* for a given interference channel $\mathbf{H}$, $\mathcal{D}_{IC}(K, [M_k], [N_j], \mathbf{H})$, is the closure of the set of all achievable DoF vectors. The DoF region $\mathcal{D}_{IC}(K, [M_k], [N_j])$ is the largest possible region such that $\mathcal{D}_{IC}(K, [M_k], [N_j]) \subset \mathcal{D}_{IC}(K, [M_k], [N_j], \mathbf{H})$

for almost all $\mathbf{H}$ in the Lebesgue sense. The *total DoF of the K-user interference channel* $\mathbf{H}$ is defined as

$$d_{IC}(K,[M_k],[N_j],\mathbf{H}) = \max_{[d_k]_{k=1}^{K} \in \mathcal{D}_{IC}(K,[M_k],[N_j],\mathbf{H})} \sum_{k=1}^{K} d_k.$$

The *total DoF* $d_{IC}(K,[M_k],[N_j])$ is defined as the largest possible real number μ such that for almost all (in the Lebesgue sense) real channel matrices $\mathbf{H}$ of size $\sum_{j\in\mathcal{K}} N_j \times \sum_{k\in\mathcal{K}} M_k$, $d_{IC}(K,[M_k],[N_j],\mathbf{H}) \geq \mu$.

Remark 1: The DoF region $\mathcal{D}_X(K,J,[M_k],[N_j])$ for the single hop wireless X network can be defined similarly as for the K-user interference channel except in this case, any DoF point in the DoF region is a matrix of the form $[d_{j,k}]_{j=1,k=1}^{J,K}$. Likewise, the DoF region $\mathcal{D}_G(K,J,[M_k],[N_j],[\mathcal{W}_j])$ for interference network with general message demand can be defined.

IV. Main Results

We develop the DoF characterizations of a variety of network communications relying on the new technique, receive antenna joint processing, and the principle of real interference alignment. The main contributions of the paper are 1) providing new proofs of Theorems 4 and 5 based on joint receive antenna processing, and 2) prove Theorems 6–8.

Theorem 4: $d_{IC}(K,[N],[N]) = \frac{NK}{2}$.

This result for constant coefficient channels has been obtained before in [4]. For time-varying channels, the same total DoF was established in [2].

Theorem 5: $d_{IC}(K,[M],[N]) \geq \frac{MN}{M+N}K$.

This result for constant coefficient channels has been obtained before in [13]. For time-varying channels, the same total DoF was established [6].

Theorem 6: The DoF region of a $(K,[M_k],[N_j])$ interference channel satisfies $\mathcal{D}_{IC}(K,[M_k],[N_j]) \supset \mathcal{D}_{IC}^{(\text{in})}$ where

$$\mathcal{D}_{IC}^{(\text{in})} := \{[d_k]_{k=1}^{K} \in \mathbb{R}_+^{K\times 1} \,\big|\, d_k + N_k \max_{\hat{k}\neq k} \frac{d_{\hat{k}}}{M_{\hat{k}}} \leq N_k, \forall k \in \mathcal{K}\}. \tag{7}$$

Corollary 1: Setting all $M_K = M$ and $N_j = N$ in Theorem 6, the DoF region of a $(K,[M],[N])$ interference channel satisfies $\mathcal{D}_{IC}(K,[M],[N]) \supset \mathcal{D}_{IC}^{(\text{in})}$ where

$$\mathcal{D}_{IC}^{(\text{in})} := \{[d_k]_{k=1}^{K} \in \mathbb{R}_+^{K\times 1} \,\big|\, M d_k + N \max_{\hat{k}\neq k} d_{\hat{k}} \leq MN, \forall k \in \mathcal{K}\}. \tag{8}$$

Corollary 2: Let assume $M = N$ in Corollary 1. Employing the outer bound derived in [5], the DoF region of a $(K, [N], [N])$ interference channel is the following

$$\mathcal{D}_{IC}(K, [N], [N]) = \{[d_k]_{k=1}^{K} \in \mathbb{R}_+^{K\times 1} \,\big|\, d_k + \max_{\hat{k}\neq k} d_{\hat{k}} \leq N, \forall k \in \mathcal{K}\}. \tag{9}$$

Theorem 7: The DoF region of a $(K, J, [N], [N], [\mathcal{W}_j])$ interference network with general message demand is

$$\mathcal{D}_G(K, J, [N], [N], [\mathcal{W}_j]) \coloneqq \{[d_k]_{k=1}^{K} \in \mathbb{R}_+^{K\times 1} \,\big|\, \sum_{k\in\mathcal{W}_j} d_k + \max_{\hat{k}\in\mathcal{W}_j^c} d_{\hat{k}} \leq N, \forall j \in \mathcal{J}\}. \tag{10}$$

Theorem 8: The DoF region of a $(K, J, [M_k], [N_j])$ X network satisfies $\mathcal{D}_X(K, J, [M_k], [N_j]) \supset \mathcal{D}_X^{(\text{in})}$ where

$$\mathcal{D}_X^{(\text{in})} \coloneqq \{[d_{j,k}]_{j=1,k=1}^{J,K} \in \mathbb{R}_+^{K\times J} \,\big|\, \sum_{k\in\mathcal{K}} d_{j,k} + N_j \sum_{j\in\mathcal{J},\hat{j}\neq j} \max_{\hat{k}\in\mathcal{K}} \frac{d_{\hat{j},\hat{k}}}{M_{\hat{k}}} \leq N_j, \forall j \in \mathcal{J}\}. \tag{11}$$

Corollary 3: As a special case of Theorem 8, the DoF region of a $(K, J, [M], [N])$ X network channel satisfies $\mathcal{D}_X(K, J, [M], [N]) \supset \mathcal{D}_X^{(\text{in})}$ where

$$\mathcal{D}_X^{(\text{in})} \coloneqq \{[d_{j,k}]_{j=1,k=1}^{J,K} \in \mathbb{R}_+^{K\times J} \,\big|\, M \sum_{k\in\mathcal{K}} d_{j,k} + N \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \max_{\hat{k}\in\mathcal{K}} d_{\hat{j},\hat{k}} \leq MN, \forall j \in \mathcal{J}\}. \tag{12}$$

Remark 2: The same DoF regions as in Corollary 2 and Theorem 7 for time-varying channel have been obtained before in [5] using vector alignment. It is interesting to note that the DoF region is regardless of whether the channel is time-varying or constant. This indicates that the DoF region for this channel is an inherent spatial property of the channel that is separate from the time or frequency diversity, as has been observed previously [5], [11].

Remark 3: Employing the outer bound derived by [10], the achieved region of Corollary 3 with the condition $M = N$ is tight in the following cases:

1) The total number of receivers is 2.
2) $d_{j,k} = d_{j,\hat{k}}$, for all $k, \hat{k} \in \mathcal{K}$ and for all $j \in \mathcal{J}$.

If we set all $d_{j,k} = \frac{N}{K+J-1}$, then we obtain the total DoF $\frac{KJN}{K+J-1}$. The same total DoF has been obtained in [11] for time-varying channel. It is again notable that the total DoF does not depend on the channel variability.

Remark 4: Let $M = 1$ in Corollary 3. In consequence, we arrive at the SIMO X network with N antenna at any receivers. For this model when $K > N$, we have the total DoF $\frac{NKJ}{K+N(J-1)}$ by fixing all $d_{j,k} = \frac{N}{K+N(J-1)}$ and the outer bound of [11]. In the rage of $K \leq N$, beamforming and zeroforcing are sufficient to achieve single-user outer bound N.

Remark 5: Theorem 4 follows from Theorem 5 by setting $M = N$ and the outer bound for K-user interference channel that has been obtained before in [2]. Moreover, Theorem 5 follows from Corollary 1 when $d_k = MN/(M+N)$, $\forall k \in \mathcal{K}$.

We conclude from the last remark that the only requirement to establish Theorem 4–5 is proving Theorem 6 (hence Corollary 1). However, we will first prove the achievability of Theorem 4 in Section V, which serves to introduce the joint antenna processing at the receivers, and the application of the result in simultaneous Diophantine approximation on manifolds. We later investigate Theorem 6–8 in succession.

V. TOTAL DOF OF $(K, [N], [N])$ INTERFERENCE CHANNEL

In this section, we examine our new achievability scheme on the $(K, [N], [N])$ interference channel. Theorem 4 is then proved by employing the outer bound in [2]. We first start with a summery of the key ideas of the achievability proof.

1) *Real interference alignment:* One important technique for proving achievability result is the real interference alignment [4] which seeks to align the dimensions of interferences so that more free dimensions can be available for intended signals. The dimensions (also named directions) are represented as real numbers that are rationally independent.
2) *Receive antenna joint processing:* We construct a coding scheme entirely based on joint signal processing at each receiver. In particular, we employ the theory of Diophantine approximation to build up the coding scheme. In order to describe more precisely, let assume a lattice generated by a $m \times n$ real matrix $\mathbf{A}$. Theorem 3 answers to the following question: how small, in terms of the size of $\mathbf{q} \in \mathbb{Z}^n$, the distance from $\mathbf{Aq}$ to $\mathbb{Z}^m$ (in particular, the all-zero vector) can be. The minimum distance proposed in the theory gives us a unique opportunity to design the coding scheme in a desired way.

Notation: We will denote the set of directions, a specific direction, and the vector of directions using $\mathcal{T}$, T, and $\mathbf{T}$ respectively.

ENCODING: Transmitter k sends a vector message $\mathbf{x}_k = (x_k^1, \ldots, x_k^N)^*$ where x_k^t, $\forall t \in \mathcal{N}$ is the signal emitted by antenna t at transmitter k. The signal x_k^t is generated using transmit directions in a set $\mathcal{T} = \{T_i \in \mathbb{R} \,\big|\, 1 \leq i \leq D\}$ as follows

$$x_k^t = \mathbf{T}\mathbf{s}_k^t \tag{13}$$

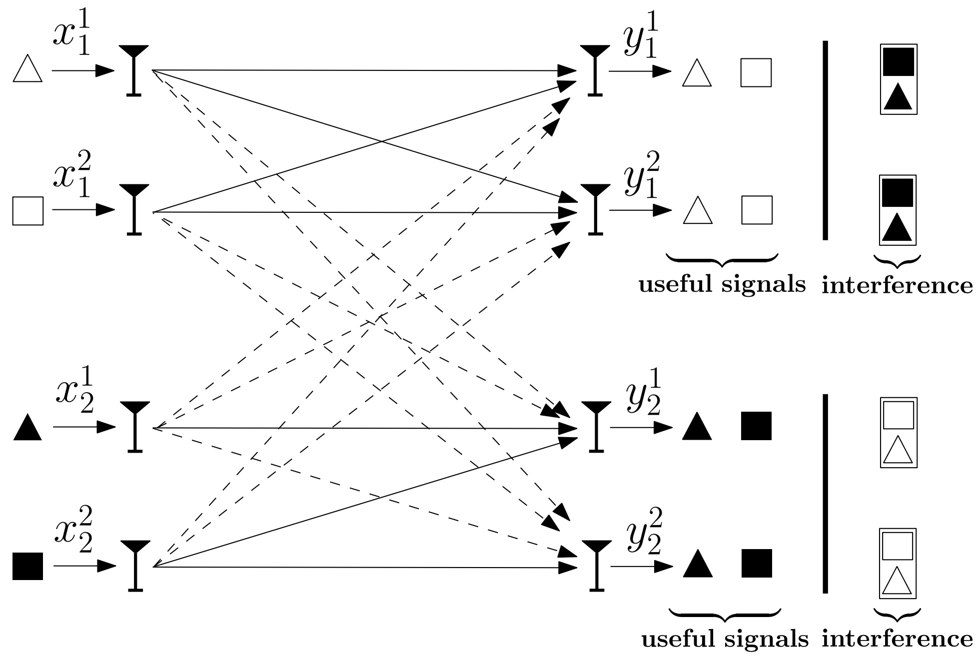

Fig. 1. 2-user Gaussian interference channel with 2 antennas at each transmitter receiver

where

$$\mathbf{T} \coloneqq (T_1, \ldots, T_D), \quad \mathbf{s}_k^t \coloneqq (s_{k1}^t, \ldots, s_{kD}^t)^* \tag{14}$$

and $\forall 1 \leq i \leq D$,

$$s_{ki}^t \in \{\lambda q \mid q \in \mathbb{Z}, -Q \leq q \leq Q\}. \tag{15}$$

The parameters Q and λ will be designed to satisfy the rate and power constraints.

ALIGNMENT DESIGN: We design transmit directions in such a way that at any receiver antenna, each useful signal occupies a set of directions that are rationally independent of interference directions.

To make it more clear, see Figure 1. Messages x_1^1 and x_1^2 are shown by white triangle and square. In a similar fashion, x_2^1 and x_2^2 are indicated with black triangle and square. We are interested in such transmit directions that at each receiver antenna the interferences, for instance black triangle and square at receiver 1, are aligned while the useful messages, white triangle and square, occupy different set of directions.

TRANSMIT DIRECTIONS: Our scheme requires all directions of set $\mathcal{T}$ to be only in the following form

$$T = \prod_{j\in\mathcal{K}} \prod_{k\in\mathcal{K}, k\neq j} \prod_{r\in\mathcal{N}} \prod_{t\in\mathcal{N}} (h_{j,k,r,t})^{\alpha_{j,k,r,t}} \tag{16}$$

where

$$0 \leq \alpha_{j,k,r,t} \leq n-1, \tag{17}$$

$\forall j \in \mathcal{K}$, $k \in \mathcal{K}$, $k \neq j$, $r \in \mathcal{N}$, $t \in \mathcal{N}$. It is easy to see that the total number directions is

$$D = n^{K(K-1)N^2}. \tag{18}$$

We also assume that directions in $\mathcal{T}$ are indexed from 1 to D. The exact indexing order is not important here.

ALIGNMENT ANALYSIS: Our design proposes that at each antenna of receiver j, $j \in \mathcal{K}$, the set of messages $\{x_k^t \,\big|\, k \in \mathcal{K}, k \neq j, t \in \mathcal{N}\}$ are aligned. To verify, consider all x_k^t, $k \neq j$ that are generated in directions of set $\mathcal{T}$. These symbols are interpreted as the interferences for receiver j. Let

$$D' = (n+1)^{K(K-1)N^2}. \tag{19}$$

and define a set $\mathcal{T}' = \{T_i' \in \mathbb{R} \,\big|\, 1 \leq i \leq D'\}$ such that all T_i' are in from of T as in (16) but with a small change as follow

$$0 \leq \alpha_{j,k,r,t} \leq n. \tag{20}$$

Clearly, all x_k^t, $k \neq j$ arrive at antenna r of receiver j in the directions of $\{(h_{j,k,r,t})\,T \,\big|\, k \in \mathcal{K}, k \neq j, t \in \mathcal{N}, T \in \mathcal{T}\}$ which is a subset of $\mathcal{T}'$.

This confirms that at each antenna of any receiver, all the interferences only contain the directions from $\mathcal{T}'$. These interference directions can be described by a vector

$$\mathbf{T}' \coloneqq (T_1', \ldots, T_{D'}'). \tag{21}$$

DECODING SCHEME: In this part, we first rewrite the received signals. Then, we prove the achievability part of Theorem 4 using joint antenna processing.

The received signal at receiver j is represented by

$$\mathbf{y}_j = \underbrace{\mathbf{H}_{j,j}\mathbf{x}_j}_{\text{the useful signal}} + \underbrace{\sum_{k \in \mathcal{K}, k \neq j} \mathbf{H}_{j,k}\mathbf{x}_k}_{\text{interference}} + \boldsymbol{\nu}_j. \tag{22}$$

Let us define

$$\mathbf{B} \coloneqq \begin{pmatrix} \mathbf{T} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{T} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{T} \end{pmatrix} \tag{23}$$

and

$$\mathbf{s}_k := \begin{pmatrix} \mathbf{s}_k^1 \\ \mathbf{s}_k^2 \\ \vdots \\ \mathbf{s}_k^N \end{pmatrix}, \quad \mathbf{u}_k := \frac{\mathbf{s}_k}{\lambda}, \tag{24}$$

such that $\mathbf{B}$ is a $N \times ND$ matrix with $(N-1)D$ zeros at each row. Using above definitions, $\mathbf{y}_j$ can be rewritten as

$$\mathbf{y}_j = \lambda \left(\mathbf{H}_{j,j}\mathbf{B}\mathbf{u}_j + \sum_{k\in\mathcal{K},k\neq j} \mathbf{H}_{j,k}\mathbf{B}\mathbf{u}_k \right) + \boldsymbol{\nu}_j. \tag{25}$$

The elements of $\mathbf{u}_k$ are integers between $-Q$ and Q, cf. (15).

We rewrite

$$\mathbf{H}_{j,j}\mathbf{B}\mathbf{u}_j = (\mathbf{H}_{j,j} \otimes \mathbf{T})\,\mathbf{u}_j = \begin{pmatrix} h_{j,j,1,1}\mathbf{T} & h_{j,j,1,2}\mathbf{T} & \ldots & h_{j,j,1,N}\mathbf{T} \\ h_{j,j,2,1}\mathbf{T} & h_{j,j,2,2}\mathbf{T} & \ldots & h_{j,j,2,N}\mathbf{T} \\ \vdots & \vdots & \ddots & \vdots \\ h_{j,j,N,1}\mathbf{T} & h_{j,j,N,2}\mathbf{T} & \ldots & h_{j,j,N,N}\mathbf{T} \end{pmatrix} \mathbf{u}_j := \begin{pmatrix} \mathbf{T}_j^1 \\ \mathbf{T}_j^2 \\ \vdots \\ \mathbf{T}_j^N \end{pmatrix} \mathbf{u}_j \tag{26}$$

where $\forall r \in \mathcal{N}$, $\mathbf{T}_j^r$ is the r^{th} row of $\mathbf{H}_{j,j}\mathbf{B}$. Also,

$$\sum_{k\in\mathcal{K},k\neq j} \mathbf{H}_{j,k}\mathbf{B}\mathbf{u}_k = \sum_{k\in\mathcal{K},k\neq j} (\mathbf{H}_{j,k} \otimes \mathbf{T})\,\mathbf{u}_k = \begin{pmatrix} \sum_{k\in\mathcal{K},k\neq j} \sum_{t\in\mathcal{N}} (h_{j,k,1,t}\mathbf{T}\mathbf{u}_k^t) \\ \sum_{k\in\mathcal{K},k\neq j} \sum_{t\in\mathcal{N}} (h_{j,k,2,t}\mathbf{T}\mathbf{u}_k^t) \\ \vdots \\ \sum_{k\in\mathcal{K},k\neq j} \sum_{t\in\mathcal{N}} (h_{j,k,N,t}\mathbf{T}\mathbf{u}_k^t) \end{pmatrix} \overset{(a)}{=} \begin{pmatrix} \mathbf{T}'\mathbf{u}_j'^1 \\ \mathbf{T}'\mathbf{u}_j'^2 \\ \vdots \\ \mathbf{T}'\mathbf{u}_j'^N \end{pmatrix} \tag{27}$$

where $\forall r \in \mathcal{N}$, $\mathbf{u}_j'^r$ is a column vector with D' integer elements, and (a) follows since the set $\mathcal{T}'$ contains all directions of the form $(h_{j,k,r,t})T$ where $k \neq j$; cf. the definition of $\mathcal{T}'$.

Considering (26) and (27), we are able to equivalently denote $\mathbf{y}_j$ as

$$\mathbf{y}_j = \lambda \begin{pmatrix} \mathbf{T}_j^1 & \mathbf{T}' & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{T}_j^2 & \mathbf{0} & \mathbf{T}' & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_j^N & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{T}' \end{pmatrix} \begin{pmatrix} \mathbf{u}_j \\ \mathbf{u}_j'^1 \\ \vdots \\ \mathbf{u}_j'^N \end{pmatrix} + \boldsymbol{\nu}_j. \tag{28}$$

It should be pointed out $\mathbf{T}_j^r$ represents the useful directions and $\mathbf{T}'$ the interference directions of antenna r at receiver j.

We finally left multiply $\mathbf{y}_j$ by an $N \times N$ weighting matrix

$$\mathbf{W} = \begin{pmatrix} 1 & \gamma_{12} & \dots & \gamma_{1N} \\ \gamma_{21} & 1 & \dots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \dots & 1 \end{pmatrix} \tag{29}$$

such that all indexed γ are randomly, independently, and uniformly chosen from interval $[\frac{1}{2}, 1]$. This process causes the zeros in (28) to be filled by non-zero directions.

After multiplying $\mathbf{W}$, the noiseless received constellation belongs to a lattice generated by the $N \times N(D+D')$ matrix

$$\mathbf{A} = \mathbf{W} \begin{pmatrix} \mathbf{T}_j^1 & \mathbf{T}' & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{T}_j^2 & \mathbf{0} & \mathbf{T}' & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_j^N & \mathbf{0} & \mathbf{0} & \dots & \mathbf{T}' \end{pmatrix}. \tag{30}$$

The above matrix has a significant property that allows us to use Theorem 3. More precisely, Theorem 3 requires each row of $\mathbf{A}$ to be a non-degenerate map from a subset of channel coefficients to $\mathbb{R}^{N(D+D')}$. The non-degeneracy is established because:

1) all elements of $\mathbf{T}'$ and $\mathbf{T}_j^t$, $\forall t \in \mathcal{N}$ are analytic functions of the channel coefficients;
2) all the directions in $\mathbf{T}'$ and $\mathbf{T}_j^t$, $\forall t \in \mathcal{N}$ together with 1 are linearly independent over $\mathbb{R}$;
3) all indexed γ in $\mathbf{W}$ have been chosen randomly and independently.

Let $\sigma(\mathbf{A})$ denote the ratio between the number of columns and the number of rows of $\mathbf{A}$. Using Theorem 3, the set of $\mathbf{H}$ such that there exist infinitely many

$$\mathbf{q} = \begin{pmatrix} \mathbf{u}_j \\ \mathbf{u}_j'^1 \\ \vdots \\ \mathbf{u}_j'^N \end{pmatrix} \in \mathbb{Z}^{N(D+D')}$$

with

$$\|\mathbf{A}\mathbf{q}\| < \|\mathbf{q}\|^{-\sigma(\mathbf{A})-\epsilon} = \|\mathbf{q}\|^{-(D+D')-\epsilon} \quad \text{for } \epsilon > 0 \tag{31}$$

has zero Lebesgue measure. In other words, for almost all $\mathbf{H}$, $\|\mathbf{A}\mathbf{q}\| > \|\mathbf{q}\|^{-(D+D')-\epsilon}$ holds for all $\mathbf{q} \in \mathbb{Z}^{N(D+D')}$ except for finite number of them. By the construction of $\mathbf{A}$, all elements in each row of $\mathbf{A}$ are rationally independent with probability one, which means that $\mathbf{A}\mathbf{q} \neq \mathbf{0}$ unless $\mathbf{q} = \mathbf{0}$. Therefore,

almost surely for any fixed channel (hence fixed $\mathbf{A}$), there is a positive constant β such that $\|\mathbf{A}\mathbf{q}\| > \beta\|\mathbf{q}\|^{-(D+D')-\epsilon}$ holds for all integer $\mathbf{q} \neq \mathbf{0}$. Since

$$\|\mathbf{q}\| \leq (K-1)NQ,$$

the distance between any two points of the received constellation (without considering noise) is lower bounded by

$$\beta\lambda\Big(\,(K-1)NQ\,\Big)^{-(D+D')-\epsilon}. \tag{32}$$

Remark 6: The noiseless received signal belongs to a constellation of the form

$$\mathbf{y} = \lambda\mathbf{A}\bar{\mathbf{q}} \tag{33}$$

where $\bar{\mathbf{q}}$ is an integer vector. Then, the decision algorithm maps the received signal to the nearest point in the constellation. Note that the decoder employs all N antennas of receiver j to detect signals emitted by intended transmitter. In other words, our decoding scheme is based on multi-antenna joint processing.

We now focus our attention on the design of λ and Q to complete the coding scheme. The parameter λ controls the input power of transmitter antennas. The average power of antenna t at transmitter k is computed as

$$P = E[{x_k^t}^2] = E[(\mathbf{T}\mathbf{s}_k^t)^2] = \sum_{i=1}^{D} {T_i}^2 E[{s_{ki}^t}^2] \leq \lambda^2 Q^2 \sum_{i=1}^{D} {T_i}^2 := \lambda^2 Q^2 \nu^2 \tag{34}$$

where the inequality follows from equation (15) and $\nu^2 := \sum_{i=1}^{D} T_i^2$. Thus, the only requirement to satisfy the power constraint is $\lambda \leq \frac{P^{\frac{1}{2}}}{Q\nu}$. To all appearances, it is sufficient to choose

$$\lambda = \frac{\zeta P^{\frac{1}{2}}}{Q}, \tag{35}$$

where $\zeta = \frac{1}{\nu}$.

Following the footsteps of [4], we assume

$$Q = P^{\frac{1-\epsilon}{2(\sigma(\mathbf{A})+1+\epsilon)}} = P^{\frac{1-\epsilon}{2(D+D'+1+\epsilon)}} \quad \text{for } \epsilon \in (0,1), \tag{36}$$

assuring that the DoF per direction at high SNR is $\frac{1-\epsilon}{D+D'+1+\epsilon}$. Since, we are allowed to arbitrarily choose ϵ within $(0,1)$, $\frac{1}{D+D'+1}$ is also achievable. We can conclude that the DoF per direction for large enough n is obtained as the number of rows over columns of matrix $\mathbf{A}$. In the next part, we verify such chosen Q and P will give us the desired performance.

PERFORMANCE ANALYSIS: Let $c_{j\,\min}$ be the minimum distance between any two points of receiver j constellation. The average error probability of confusing s_{ki}^t with $\hat{s}_{ki}^t$ (detected message) when s_{ki}^t is transmitted is upper bounded by

$$P_e = Pr(s_{ki}^t \to \hat{s}_{ki}^t) \leq \exp\left(\frac{-c_{j\,\min}^2}{8}\right) \tag{37}$$

for all $k \in \mathcal{K}$, $t \in \mathcal{N}$, $1 \leq i \leq D$. According to (32), $c_{j\,\min}$ is bounded as follow

$$c_{j\,\min} > \beta\lambda\big(\,(K-1)NQ\,\big)^{-(D+D')-\epsilon}. \tag{38}$$

Substituting (35) and (36) in above inequality, we arrive at

$$c_{j\,\min} > \kappa P^{\frac{\epsilon}{2}} \tag{39}$$

with constant $\kappa = \beta\zeta\big(\,(K-1)N\,\big)^{-(D+D')-\epsilon}$. Therefore, the hard decoding error probability of received constellation can be represented as

$$P_e < \exp\big(\frac{-\kappa^2}{8}P^{\epsilon}\big), \tag{40}$$

which goes to zero as $P \to \infty$. For this performance, the total achievable DoF for almost all channel coefficients in the Lebesgue sense is

$$\frac{NKD}{D+D'+1} = \frac{NKn^{K(K-1)N^2}}{n^{K(K-1)N^2} + (n+1)^{K(K-1)N^2} + 1} \tag{41}$$

and as n increases, the total DoF goes to $\frac{NK}{2}$ which meets the outer bound [2].

VI. K-USER INTERFERENCE CHANNEL AND INNER BOUND ON DOF REGION

Notation: Unless otherwise stated, all the assumptions and definitions are still the same for the next sections.

For sake of simplicity, we first prove Corollary 1 in this section. Then utilizing the presented proof, Theorem 6 is easily established.

Consider a $(K, [M], [N])$ MIMO interference channel. We prove that for any $[d_k]_{k=1}^K \in \mathcal{D}_{IC_1}^{(\text{in})}$, $[d_k]_{k=1}^K$ is achievable.

Under the rational assumption, it is possible to find an integer ρ such that $\forall k \in \mathcal{K}$, $\bar{d}_k = \rho\frac{d_k}{M}$ is a non-negative integer. The signal x_k^t is divided into $\bar{d}_k$ streams. For stream l, $l \in \{1, \ldots, \max_{k\in\mathcal{K}} \bar{d}_k\}$, we use directions $\{T_{l1}, \ldots, T_{lD}\}$ of the following form

$$T_l = \prod_{j\in\mathcal{K}} \prod_{k\in\mathcal{K}, k\neq j} \prod_{r\in\mathcal{M}} \prod_{t\in\mathcal{N}} (h_{j,k,r,t}\delta_l)^{\alpha_{j,k,r,t}} \tag{42}$$

where $0 \leq \alpha_{j,k,r,t} \leq n-1$ and δ_l is a design parameter that is chosen randomly, independently, and uniformly from the interval $[\frac{1}{2}, 1]$. Let $\mathbf{T}_l \coloneqq (T_{l1}, \ldots, T_{lD})$. Note that, at any antenna of transmitter k, the constants $\{\delta_l\}$ cause the streams to be placed in $\bar{d}_k$ different sets of directions. The alignment scheme is the same as before, considering the fact that at each antenna of receiver j, the useful streams occupy $M\bar{d}_j$ separate sets of directions. The interferences are also aligned at most in $\max\limits_{k\in\mathcal{K},k\neq j} \bar{d}_k$ sets of directions independent from useful directions.

By design, x_k^t is emitted in the following form

$$x_k^t = \sum_{l=1}^{\bar{d}_k} \delta_l \sum_{i=1}^{D} T_{li} s_{kli}^t = \mathbf{T}_k \mathbf{s}_k^t \tag{43}$$

where

$$\mathbf{T}_k \coloneqq (\delta_1 \mathbf{T}_1, \ldots, \delta_{\bar{d}_k} \mathbf{T}_{\delta_{\bar{d}_k}}), \tag{44}$$

$$\mathbf{s}_k^t \coloneqq (s_{k11}^t, \ldots, s_{k\bar{d}_k D}^t)^*, \tag{45}$$

and all s_{kli}^t belong to the set defined in (15).

Pursuing the same steps of the previous section for receiver j, $\mathbf{B}$ becomes a $M \times MD\bar{d}_j$ matrix as

$$\begin{pmatrix} \mathbf{T}_j & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_j & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{T}_j \end{pmatrix} \tag{46}$$

and $\mathbf{A}$ will have N rows and $MD\bar{d}_j + ND' \max\limits_{k\in\mathcal{K},k\neq j} \bar{d}_k$ columns. To be more precise, matrix $\mathbf{A}$ has the same form as (30) noting that $\mathbf{T}_j^r$ and $\mathbf{T}'$ are now vectors with $M\bar{d}_j$ and $\max\limits_{k\in\mathcal{K},k\neq j} \bar{d}_k$ elements, respectively.

Remark 7: As it has been proved in the previous section, the dimensions of matrix $\mathbf{A}$ inherits two momentous characteristics as follows:

1) The number of columns shows all available directions at the receiver.
2) For large n, the number of rows over columns specifies the achievable DoF per direction.

The total number directions G_j of both useful signals and the interferences at receiver j can not be greater than all available directions. This results in

$$G_j \leq MD\bar{d}_j + ND' \max_{k\in\mathcal{K},k\neq j} \bar{d}_k. \tag{47}$$

For any DoF points in $\mathcal{D}_{IC}^{(\text{in})}$ satisfying Corollary 1, we have

$$G_j \leq \left(M\bar{d}_j + N \max_{k\in\mathcal{K},k\neq j} \bar{d}_k \right) D' \leq \frac{\rho}{M} NMD' = \rho ND' \tag{48}$$

and as n increases, the DoF of the signal $\mathbf{x}_j$ intended for receiver j, $\forall j \in \mathcal{K}$ can be arbitrarily close to

$$\lim_{n\to\infty} MD\bar{d}_j \frac{N}{\rho N D'} = \lim_{n\to\infty} \frac{M}{\rho} \frac{\bar{d}_j n^{K(K-1)N^2}}{(n+1)^{K(K-1)N^2}} = \frac{M}{\rho}\bar{d}_j = d_j \tag{49}$$

where $\frac{N}{\rho N D'}$ is the DoF per direction for large D'. This proves Theorem 1.

As a special case, it is easy to see when all d_k are equal, the total achievable DoF is $\frac{MN}{M+N}K$. Moreover, when $M = N$, the achievable DoF region is tight, cf. Remark 9.

To establish Theorem 6, we follow the proof of Corollary 1 with a small change in assumption, which is $\bar{d}_k = \rho\frac{d_k}{M_k}$. As a result, $\mathbf{A}$ becomes N_j by $M_k D\bar{d}_j + N_j D' \max_{k\in\mathcal{K}, k\neq j} \bar{d}_k$ matrix. Therefore, for any DoF points in $\mathcal{D}_{IC}^{(\text{in})}$ satisfying Theorem 6, we have

$$G_j \leq M_K D\bar{d}_j + N_j D' \max_{k\in\mathcal{K}, k\neq j} \bar{d}_k \leq \rho N_j D' \tag{50}$$

and the DoF of signal x_j is finally obtained as

$$\lim_{n\to\infty} M_k D\bar{d}_j \frac{N_j}{\rho N_j D'} = \lim_{n\to\infty} d_j \frac{n^{K(K-1)N^2}}{(n+1)^{K(K-1)N^2}} = d_j. \tag{51}$$

VII. Interference network with general message demands

Consider a $(K, J, [N], [N], [\mathcal{W}_j])$ single hop interference network with general message demand. Transmitter k emits independent message $\mathbf{x}_k$, and receiver j requests an arbitrary subset of messages denoted by $\mathcal{W}_j$. We follow the same definitions and steps of Section VI considering stream l, uses directions of the following form

$$T_l = \prod_{j\in\mathcal{J}} \prod_{k\in\mathcal{W}_j^c} \prod_{r\in\mathcal{N}} \prod_{t\in\mathcal{N}} (h_{j,k,r,t}\delta_l)^{\alpha_{j,k,r,t}} \tag{52}$$

where $0 \leq \alpha_{j,k,r,t} \leq n-1$, $\mathcal{W}_j^c := \{k \in \mathcal{K} \,\big|\, k \notin \mathcal{W}_j\}$, and δ_l is a design parameter chosen as before. Notice that the directions has been designed in such a manner that at any receiver, for example receiver j, while the useful signal subspace is separated from the interference subspace, all interferences caused by $\mathbf{x}_k$, $k \in \mathcal{W}_j$ are aligned. As a result, matrix $\mathbf{A}$ at receiver j will have N rows and $ND\sum_{k\in\mathcal{W}_j} \bar{d}_k + ND' \max_{\hat{k}\in\mathcal{W}_j^c} \bar{d}_{\hat{k}}$ columns. Thus, for any DoF point in $\mathcal{D}_G^{(\text{in})}$ satisfying Theorem 7, G_j is upper bounded by $\rho N D'$ and d_k, $k \in \mathcal{W}_j$, is achieved similar to (49).

VIII. WIRELESS X NETWORKS

Consider a $(K, J, [M], [N])$ Gaussian X network. For each pair $(j, k) \in \mathcal{J} \times \mathcal{K}$, transmitter k sends a $M \times 1$ vector message $\mathbf{x}_{j,k} = (x_{j,k}^1, \dots, x_{j,k}^M)^*$ to receiver j. Consequently, the signal emitted by transmitter k is in the following form

$$\mathbf{x}_k = \sum_{j \in \mathcal{J}} \mathbf{x}_{j,k}. \tag{53}$$

By rational assumption, we can find an integer ρ such that for all $j \in \mathcal{J}$ and all $k \in \mathcal{K}$, $\bar{d}_{j,k} = \rho \frac{d_{j,k}}{M}$ is a non-negative integer. Message $x_{j,k}^t$ is divided into $\bar{d}_{j,k}$ streams such that each stream, say stream $l \in \{1, \dots, \max\limits_{k \in \mathcal{K}} \bar{d}_{j,k}\}$, uses directions in set $\mathcal{T}_{j,l} = \{T_{j,l,i} \in \mathbb{R} \,\big|\, 1 \leq i \leq D\}$. All $T_{j,l,i}$ are generated in the following form

$$T_{j,l} = \prod_{\hat{j} \in \mathcal{J}, \hat{j} \neq j} \prod_{\hat{k} \in \mathcal{K}} \prod_{\hat{r} \in \mathcal{N}} \prod_{\hat{t} \in \mathcal{M}} \left(h_{\hat{j}, \hat{k}, \hat{r}, \hat{t}} \delta_{j,l} \right)^{\alpha_{\hat{j}, \hat{k}, j, \hat{r}, \hat{t}, l}} \tag{54}$$

where $0 \leq \alpha_{\hat{j}, \hat{k}, j, \hat{r}, \hat{t}, l} \leq n - 1$ and $\delta_{j,l}$ is a design parameter that is chosen randomly, independently, and uniformly from the interval $[\frac{1}{2}, 1]$. Define $\mathbf{T}_{j,l} := (T_{j,l,1}, \dots, T_{j,l,D})$. The signal $x_{j,k}^t$ is generated as

$$x_{j,k}^t = \sum_{l=1}^{\bar{d}_{j,k}} \delta_{j,l} \sum_{i=1}^{D} T_{j,l,i} s_{j,k,l,i}^t = \mathbf{U}_{j,k} \mathbf{s}_{j,k}^t \tag{55}$$

where

$$\mathbf{U}_{j,k} = (\delta_{j,1} \mathbf{T}_{j,1}, \dots, \delta_{j,\bar{d}_{j,k}} \mathbf{T}_{j,\bar{d}_{j,k}}), \tag{56}$$

$$\mathbf{s}_{j,k}^t = (s_{j,k,1,1}^t, \dots, s_{j,k,\bar{d}_{j,k},D}^t)^*, \tag{57}$$

and all $s_{j,k,l,i}^t$ are members of the set in (15).

ALIGNMENT DESIGN: Suppose we are at receiver j. The design of transmit directions guarantees that at any antenna of receiver j, the useful signals are placed in K separate sets of directions. Each set has $D\bar{d}_{j,k}$, $k \in \mathcal{K}$ directions. The interferences are also put in $J - 1$ different sets of directions, each containing all signals intended for receiver $\hat{j}$, $\hat{j} \in \mathcal{J}$, $\hat{j} \neq j$ with at most $D' \max\limits_{k \in \mathcal{K}} \bar{d}_{\hat{j},k}$ directions.

Let us explain the above mentioned argument for a $(3, 3, [1], [2])$ Gaussian X network. This system is depicted in Figure 2. Each transmitter conveys an independent message to each receiver. We have assumed that white square, triangle, and circle are the useful signals for the first receiver. Similarly, black and gray colors show the signals intended for receiver 2 and 3, respectively. The transmission scheme is such that at any antenna of receiver 1:

- The interferences, black square triangle and circle, are aligned. The gray signals are also aligned.

- The useful signals, white square triangle and circle, are not aligned.

Hence, at each receive antenna of first user, we have the sum of five terms made by three useful signals and two set of aligned signals. The set of directions used for each term is separate from others in sense of rational independence. A similar statement is also valid for other receivers. We prove Theorem 3 provided that the described alignment scheme is successful.

ALIGNMENT VERIFICATION: The proposed transmit directions guarantee that the interferences created by messages intended for the same receiver are aligned at all other receivers. To see this, let us define $\mathcal{T}'_{j,l} = \{T'_{j,l,i} \in \mathbb{R} \,\big|\, 1 \leq i \leq D'\}$ such that all $T'_{j,l,i}$ are in the form of (54) but with $0 \leq \alpha_{\hat{j},\hat{k},j,\hat{r},\hat{t},l} \leq n$. We use $\mathbf{T}'_{j,l}$ to denote vector $(T'_{j,l,1}, \ldots, T'_{j,l,D'})$. According to (55), the l^{th} stream of message $x^t_{j,k}$ is transmitted in directions of the form $\delta_{j,l}T_{j,l}$. This stream arrives at antenna r of receiver $\hat{j}$, $\hat{j} \neq j$, in directions of the form $\left(h_{\hat{j},k,r,t}\delta_{j,l}\right) T_{j,l}$, which are obviously in set $\mathcal{T}'_{j,l}$. Since $\mathcal{T}'_{j,l}$ does not depend on indices k and r, cf. (54), at any antenna of receiver $\hat{j}$, $\hat{j} \neq j$, all directions created by the streams intended for receiver j are subset of $\mathcal{T}'_{j,l}$, $\forall l \in \{1, \ldots, \max_{k\in\mathcal{K}} \bar{d}_{j,k}\}$ and occupy at most $D' \max_{k\in\mathcal{K}} \bar{d}_{j,k}$ dimensions. We denote these directions as a vector $\mathbf{T}'_j := (\mathbf{T}'_{j,1}, \ldots, \mathbf{T}'_{j,\max_{k\in\mathcal{K}} \bar{d}_{j,k}})$.

DECODING SCHEME: The received signal at receiver j can be divided into two parts, the useful signals and interference, of the following form

$$\mathbf{y}_j = \sum_{k\in\mathcal{K}} \mathbf{H}_{j,k}\mathbf{x}_{j,k} + \sum_{k\in\mathcal{K}} \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \mathbf{H}_{\hat{j},k}\mathbf{x}_{\hat{j},k} + \boldsymbol{\nu}. \tag{58}$$

For notational convenience, let $\mathbf{s}_{j,k} := (\mathbf{s}^1_{j,k}, \ldots, \mathbf{s}^M_{j,k})^*$ and $\mathbf{u}_j := \frac{1}{\lambda}(\mathbf{s}_{j,1}, \ldots, \mathbf{s}_{j,K})^*$ with integer elements between $-Q$ and Q. Then, we can rewrite the useful signals as follows

$$\sum_{k\in\mathcal{K}} \mathbf{H}_{j,k}\mathbf{x}_{j,k} = \sum_{k\in\mathcal{K}} \mathbf{H}_{j,k} \begin{pmatrix} x^1_{j,k} \\ x^2_{j,k} \\ \vdots \\ x^M_{j,k} \end{pmatrix} \overset{(b)}{=} \sum_{k\in\mathcal{K}} \begin{pmatrix} h_{j,k,1,1}\mathbf{U}_{j,k} & h_{j,k,1,2}\mathbf{U}_{j,k} & \ldots & h_{j,k,1,N}\mathbf{U}_{j,k} \\ h_{j,k,2,1}\mathbf{U}_{j,k} & h_{j,k,2,2}\mathbf{U}_{j,k} & \ldots & h_{j,k,2,N}\mathbf{U}_{j,k} \\ \vdots & \vdots & \ddots & \vdots \\ h_{j,k,N,1}\mathbf{U}_{j,k} & h_{j,k,N,2}\mathbf{U}_{j,k} & \ldots & h_{j,k,N,N}\mathbf{U}_{j,k} \end{pmatrix} \mathbf{s}_{j,k} \tag{59}$$

$$:= \sum_{k\in\mathcal{K}} \begin{pmatrix} \mathbf{U}^1_{j,k} \\ \mathbf{U}^2_{j,k} \\ \vdots \\ \mathbf{U}^N_{j,k} \end{pmatrix} \mathbf{s}_{j,k} = \lambda \begin{pmatrix} \mathbf{U}^1_{j,1} & \mathbf{U}^1_{j,2} & \ldots & \mathbf{U}^1_{j,K} \\ \mathbf{U}^2_{j,1} & \mathbf{U}^2_{j,2} & \ldots & \mathbf{U}^2_{j,K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{U}^N_{j,1} & \mathbf{U}^N_{j,2} & \ldots & \mathbf{U}^N_{j,K} \end{pmatrix} \mathbf{u}_j \tag{60}$$

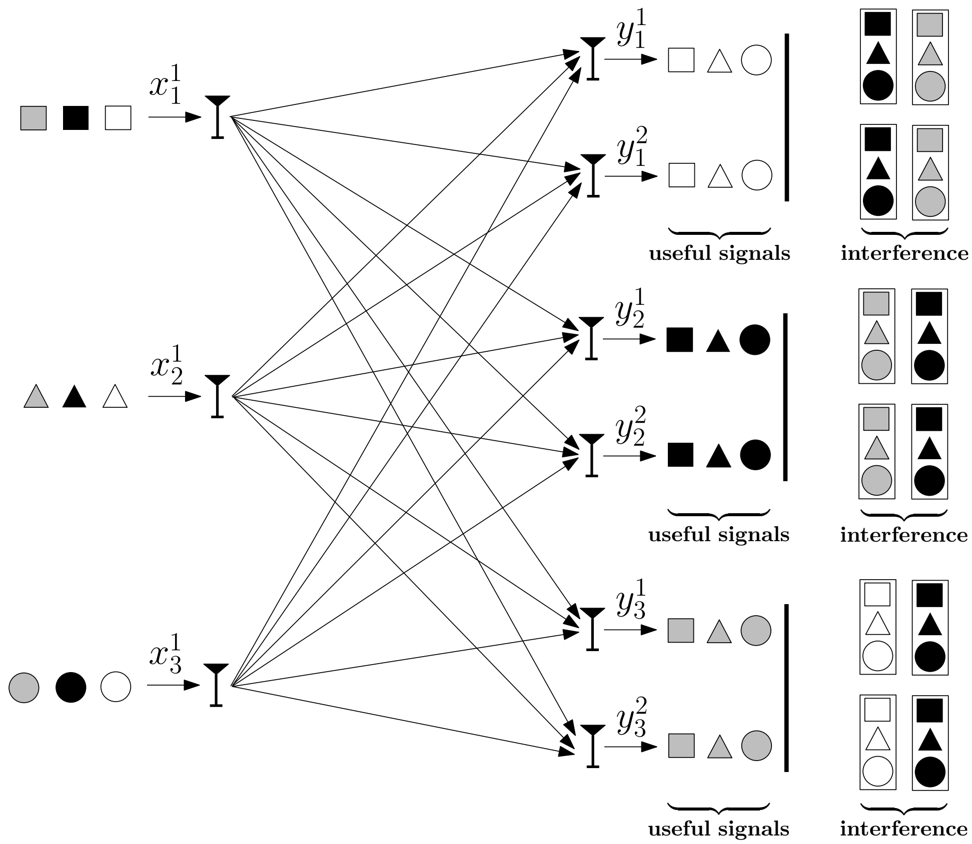

Fig. 2. $(3 \times 3, 1, 2)$ Gaussian X network channel

where $\mathbf{U}_{j,k}^{r} \coloneqq (h_{j,k,r,1}\mathbf{U}_{j,k}, h_{j,j,r,2}\mathbf{U}_{j,k}, \ldots, h_{j,j,r,N}\mathbf{U}_{j,k})$, $\forall j \in \mathcal{J}$, $k \in \mathcal{K}$, $r \in \mathcal{N}$. Using the definition in (55), (b) follows. We take into account that none of $\mathcal{T}'_{\hat{j},l}$, $\hat{j} \neq j$, contains generators $\{(h_{j,k,r,t}\delta_{j,l}) \,\big|\, k \in \mathcal{K}, r \in \mathcal{N}, t \in \mathcal{M}\}$. Hence, the directions in all $\mathbf{U}_{j,k}^{r}$ and $\mathbf{T}'_{\hat{j}}$, $\hat{j} \neq j$ are rationally independent.

The interference part can be written as

$$\begin{aligned}
\sum_{k\in\mathcal{K}} \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \mathbf{H}_{\hat{j},k}\mathbf{x}_{\hat{j},k} &= \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \sum_{k\in\mathcal{K}} \mathbf{H}_{\hat{j},k} \begin{pmatrix} x_{\hat{j},k}^{1} \\ x_{\hat{j},k}^{2} \\ \vdots \\ x_{\hat{j},k}^{M} \end{pmatrix} \overset{(c)}{=} \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \sum_{k\in\mathcal{K}} \mathbf{H}_{\hat{j},k} \begin{pmatrix} \mathbf{U}_{\hat{j},k}\mathbf{s}_{\hat{j},k}^{1} \\ \mathbf{U}_{\hat{j},k}\mathbf{s}_{\hat{j},k}^{2} \\ \vdots \\ \mathbf{U}_{\hat{j},k}\mathbf{s}_{\hat{j},k}^{M} \end{pmatrix} \\
&= \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \begin{pmatrix} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{M}} \left(h_{\hat{j},k,1,t}\mathbf{U}_{\hat{j},k}\mathbf{s}_{\hat{j},k}^{t}\right) \\ \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{M}} \left(h_{\hat{j},k,2,t}\mathbf{U}_{\hat{j},k}\mathbf{s}_{\hat{j},k}^{t}\right) \\ \vdots \\ \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{M}} \left(h_{\hat{j},k,N,t}\mathbf{U}_{\hat{j},k}\mathbf{s}_{\hat{j},k}^{t}\right) \end{pmatrix} \overset{(d)}{=} \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \lambda \begin{pmatrix} \mathbf{T}'_{\hat{j}}\mathbf{u}'^{1}_{\hat{j}} \\ \mathbf{T}'_{\hat{j}}\mathbf{u}'^{2}_{\hat{j}} \\ \vdots \\ \mathbf{T}'_{\hat{j}}\mathbf{u}'^{N}_{\hat{j}} \end{pmatrix}
\end{aligned} \tag{61}$$

where for all $r \in \mathcal{N}$, $\mathbf{u}'^{r}_{\hat{j}}$ is a column vector with integer elements. Equivalence relation (c) follows from

(55). The equality (d) is due to alignment by our design. It is convenient to represent equation (61) as

$$\lambda \begin{pmatrix} \mathbf{I}_j \mathbf{z}_j^1 \\ \mathbf{I}_j \mathbf{z}_j^2 \\ \vdots \\ \mathbf{I}_j \mathbf{z}_j^N \end{pmatrix} \tag{62}$$

where $\mathbf{I}_j \coloneqq (\mathbf{T}'_1, \ldots, \mathbf{T}'_{j-1}, \mathbf{T}'_{j+1}, \ldots, \mathbf{T}'_J)$ and $\mathbf{z}_j^r \coloneqq (\mathbf{u}'^r_1, \ldots, \mathbf{u}'^r_{j-1}, \mathbf{u}'^r_{j+1}, \ldots, \mathbf{u}'^r_J)$ for all $t \in \mathcal{N}$.

Using (60) and (62), received signal $\mathbf{y}_j$ is represented by

$$\lambda \begin{pmatrix} \mathbf{U}_{j,1}^1 & \mathbf{U}_{j,2}^1 & \ldots & \mathbf{U}_{j,K}^1 & \mathbf{I}_j & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{U}_{j,1}^2 & \mathbf{U}_{j,2}^2 & \ldots & \mathbf{U}_{j,K}^2 & \mathbf{0} & \mathbf{I}_j & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{U}_{j,1}^N & \mathbf{U}_{j,2}^N & \ldots & \mathbf{U}_{j,K}^N & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I}_j \end{pmatrix} \begin{pmatrix} \mathbf{u}_j \\ \mathbf{z}_j^1 \\ \vdots \\ \mathbf{z}_j^N \end{pmatrix} + \boldsymbol{\nu}_j. \tag{63}$$

Analogous to achievability proof of Theorem 4, we left multiply $\mathbf{y}_j$ by an $N \times N$ weighting matrix. Then, $\mathbf{A}$ in (30) becomes a $N \times (MD\sum_{k\in\mathcal{K}} \bar{d}_{j,k} + ND'\sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \max_{k\in\mathcal{K}} \bar{d}_{\hat{j},k})$ matrix with the same characteristics as before.

The total directions G_j of the useful signals and the interferences at receiver j is

$$G_j \leq MD\sum_{k\in\mathcal{K}} \bar{d}_{j,k} + ND' \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \max_{k\in\mathcal{K}} \bar{d}_{\hat{j},k}. \tag{64}$$

Consequently, any DoF point in $\mathcal{D}_{XC_1}^{(\text{in})}$ that satisfies Theorem 3 has $G_j \leq \rho N D'$. Thus, as n increases, the DoF of $\mathbf{x}_{j,k}$, $\forall j \in \mathcal{J}$, $k \in \mathcal{K}$, goes to

$$\lim_{n\to\infty} MD\bar{d}_{j,k}\frac{N}{\rho ND'} = \lim_{n\to\infty} \frac{M}{\rho}\frac{\bar{d}_{j,k} n^{K(K-1)N^2}}{(n+1)^{K(K-1)N^2}} = \frac{M}{\rho}\bar{d}_{j,k} = d_{j,k}, \tag{65}$$

which establishes Theorem 3.

The provided scheme for the $(K, J, [M], [N])$ Gaussian X network channel can be applied to a more general case where each transmitter/receiver has arbitrary number of antennas. Let us assume that transmitter k has M_k antennas and receiver j has N_j antennas. To prove Theorem 8, we follow the same procedure of this section for receiver j considering the integer ρ is changed such that $\bar{d}_{j,k} = \rho\frac{d_{j,k}}{M_k}$, $\forall k \in \mathcal{K}$, $j \in \mathcal{J}$. Accordingly, $\mathbf{A}$ becomes a $N \times (D\sum_{k\in\mathcal{K}} M_k\bar{d}_{j,k} + N_jD'\sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \max_{k\in\mathcal{K}} \bar{d}_{\hat{j},k})$ matrix. Hence, the total number of useful and interference directions at receiver j is

$$G_j \leq D\sum_{k\in\mathcal{K}} M_k\bar{d}_{j,k} + N_jD' \sum_{\hat{j}\in\mathcal{J},\hat{j}\neq j} \max_{k\in\mathcal{K}} \bar{d}_{\hat{j},k} \tag{66}$$

and $G_j \leq \rho N_j D'$ for any DoF point in $\mathcal{D}_{XC_2}^{(\text{in})}$ satisfying Theorem 8. As a result, for large enough n, the DoF of signal $\mathbf{x}_{j,k}$ is attained as

$$\lim_{n\to\infty} M_k D\bar{d}_{j,k}\frac{N_j}{\rho N_j D'} = \lim_{n\to\infty}\frac{M_k}{\rho}\frac{\bar{d}_{j,k}n^{K(K-1)N^2}}{(n+1)^{K(K-1)N^2}} = \frac{M_k}{\rho}\bar{d}_{j,k} = d_{j,k} \tag{67}$$

for all $j \in \mathcal{J}$ and $k \in \mathcal{K}$. This completes the proof.

IX. Outer Bound Discussion

Although our main focus in this paper is on the new receive antenna joint processing, we present a brief discussion on existing outer bounds of interference networks. Note that all outer bounds are completely general as it applies to interference networks regardless of whether the channel coefficients are time varying or constant.

Ghasemi et al. in [7] show that the total DoF of $(K,[M],[N])$ MIMO Gaussian interference channel is outer bounded by $K\frac{MN}{M+N}$ when $K \geq \frac{M+N}{\text{gcd}(M,N)}$. To establish this result, first consider a $(L,[M],[N])$ MIMO interference channel where $L \leq K$. For this scenario, the L users are divided into two arbitrary disjoint sets of size L_1 and L_2 such that $L = L_1 + L_2$. The full cooperation among transmitters in each set is assumed and similarly for each set of receivers. Accordingly, the 2-user MIMO interference channel with L_1M, L_2M antenna at transmitters and L_1N, L_2N antennas at receivers is obtained. Using the DoF region of 2-user MIMO interference channel [8], the DoF is finally outer bounded.

It is also shown that for $K \leq \frac{\max(M,N)}{\min(M,N)}+1$, the total DoF outer bound is $\min(M,N)\min(K,\frac{\max(M,N)}{\min(M,N)})$. However, the DoF characterization for the remaining region $\lfloor\frac{\max(M,N)}{\min(M,N)}\rfloor+1 < K < \frac{M+N}{\text{gcd}(M,N)}$ has not been established due to the complexity of convex optimizations over integers. To understand the origin of this problem, we next examine the mentioned scheme when L_2M has the minimum difference from L_1N and we extend the result to obtain an outer bound on the DoF region.

The key to establishing the outer bound on $(K,[M],[N])$ interference channel is to consider a set of g receivers as a group. For this receiver set, the corresponding transmitters emitting useful signals are assumed to be cooperative as one set. Hence, the rest of transmitters only create interference. We then pick a subset of the remaining transmitters such that their total number of antennas is the closest to the number of antennas of the receiver set, namely gN. Such grouping creates a two users MIMO interference channel to which the known DoF region will be applied.

Consider an arbitrary subset of receivers $G_{R_1} \subseteq \mathcal{K}$ with cardinality g. Let $G_{T_1} = G_{R_1}$. The set G_{T_1} contains indices of transmitters whose signals are useful for the receivers in G_{R_1}. We define another subset

of transmitters, $G_{T_2} \subseteq \mathcal{K} \setminus G_{T_1}$, such that

1) The cardinality of G_{T_2} is $\min\{K-g, \lfloor \frac{gN}{M} \rfloor\}$.
2) Set G_{T_2} maximizes $\sum_{k \in G_{T_2}} d_k$.

The corresponding receivers of G_{T_2} are shown by set G_{R_2}. We then remove all the remaining users with indices in $\mathcal{K} \setminus \{G_{T_1} \cup G_{T_2}\}$.

In [8], it is proved that the DoF region for a 2-user MIMO Gaussian interference channel with M_1, M_2 antennas at transmitters and N_1, N_2 antennas at the corresponding receivers is

$$\begin{aligned} d_1 \leq \min(M_1, N_1), \quad d_2 \leq \min(M_2, N_2) \\ d_1 + d_2 \leq \min\{M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_1)\} \end{aligned} \tag{68}$$

Using above result when G_{T_1}, G_{R_1} are viewed as the first user and G_{T_2}, G_{R_2} as the second user, we obtain

$$\sum_{k \in G_{T_1}} d_k \leq g \min(M, N) \tag{69}$$

$$\sum_{\hat{k} \in G_{T_2}} d_{\hat{k}} \leq \min\{K-g, \lfloor \frac{gN}{M} \rfloor\} \min(M, N) \tag{70}$$

$$\sum_{k \in G_{T_1}} d_k + \sum_{\hat{k} \in G_{T_2}} d_{\hat{k}} \leq gN. \tag{71}$$

Given g and G_{T_1}, equations (69)–(71) define a DoF region outer bound. By considering all $1 \leq g \leq K$ and for each g all possible $G_{T_1} \subseteq \mathcal{K}$ with cardinality g, the outer bound can be optimized.

As a special case, if we set all d_k equal to d, we have

$$gd + \min\{K-g, \lfloor \frac{gN}{M} \rfloor\} d \leq gN \tag{72}$$

for all $g \in \mathcal{K}$. The above inequality can be represented as

$$d \leq \frac{gN}{\min\{K, \lfloor \frac{g(N+M)}{M} \rfloor\}}. \tag{73}$$

Therefore, the outer bound for the total DoF is obtained as

$$\min_{g \in \mathcal{K}} \frac{gNK}{\min\{K, \lfloor \frac{g(N+M)}{M} \rfloor\}}. \tag{74}$$

For $K \geq \frac{M+N}{\text{gcd}(M,N)}$, we are able to choose $g = \frac{M}{\text{gcd}(M,N)}$ resulting in the same number of antennas at transmitters k, $\forall k \in G_{T_2}$, and receivers j, $\forall j \in G_{R_1}$. Subsequently, the total DoF is upper bounded by $\frac{MN}{M+N} K$, which is achievable according to Theorem 5.

As it appears, having an identical number of antennas at the receive side of user 1 and transmit side of user 2 is important for establishing the optimality of total DoF. In other words, the desired outer bound occurs when the receivers of group user 1 with gN antennas are able to successfully decode interferences created by gN antennas. Such requirement can be satisfied if $K \geq \frac{M+N}{\gcd(M,N)}$.

Remark 8: Zero-forcing always allows us to achieve the total DoF $\min\{\max(M,N), K\min(M,N)\}$, which is indeed tight when $K < \frac{M+N}{\min(M,N)}$, cf. [7].

Remark 9: In the case $M = N$, the best g is one. Therefore, the DoF region is upper bounded by

$$d_k + \max_{\hat{k}\in\mathcal{K}, \hat{k}\neq k} d_{\hat{k}} \leq N \tag{75}$$

for all $k \in \mathcal{K}$.

To improve outer bounds associated with grouping approach, a new method in [9] called genie chains is proposed where a receiver is provided with a subspace of signals (part of transmitted symbols) as a genie. As a result of this approach, the total DoF $\frac{MN}{M+N}$ is obtained for the wider range of $\frac{M}{N} \geq \frac{K-2}{K^2-3K+1}$.

In MIMO X network channel, the only well known outer bound has been determined by [10]. It is shown that the sum of all the DoFs of the messages associated with transmitter k and receiver j is upper bounded by $\max(M_k, N_j)$. Despite the assurance that the total DoF outer bound is achieved for the single antenna X network, the characterization for the case of MIMO seems to be challenging.

X. Conclusions and Future Works

We developed a new real interference alignment scheme for multiple-antenna interference networks that employs joint receiver antenna processing. The scheme utilized a result on simultaneous Diophantine approximation and aligned all interferences at each receive antenna. We were able to derive several new DoF region results (Theorems 6–8).

Moreover, it was proved that the total DoF of constant wireless X networks with N antennas at each node is tight. This showed us coding schemes based on cooperation among receive antennas is important for establishing the optimal total DoF, as previously observed by [11] in the case of time-varying channels.

It is desired to extend the result of the paper to a multiple-antenna interference network with K transmitters and J receivers where each transmitter sends an arbitrary number of messages, and each receiver may be interested in an arbitrary subset of the transmitted messages.

It is also possible that one can improve the existing outer bounds so that the optimality of the achieved DoF regions are generally proved.

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